RAPPELS :

Forme Canonique : $a(x+\alpha)^2+\beta$

Équation du second degré : $ax^2+bx+c=0$

Discriminant : $\Delta = b^2-4ac$

Les racines : $x_1 = \dfrac{-b+\sqrt{\Delta}}{2a}$ $\quad x_2 = \dfrac{-b-\sqrt{\Delta}}{2a}$

La somme des racines : $S = \dfrac{-b}{a}$

Le produit des racines : $P = \dfrac{c}{a}$

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1) a) $f(x) = x^2-\sqrt{2}x-4 = \left(x-\frac{\sqrt{2}}{2}\right)^2-\frac{1}{2}-\frac{8}{2}$

$$= \left(x-\frac{\sqrt{2}}{2}\right)^2-\frac{9}{2}$$

b) $g(x) = 6x^2+x-2 = 6\left(x^2+\frac{x}{6}-\frac{1}{3}\right)$

$$= 6\left[\left(x+\frac{1}{12}\right)^2-\frac{1}{144}-\frac{48}{144}\right]$$

$$= 6\left(x+\frac{1}{12}\right)^2-\frac{49}{24}$$

c) $h(x) = 3x^2-20x+12 = 3\left(x^2-\frac{20}{3}x+4\right)$

$$= 3\left[\left(x-\frac{10}{3}\right)^2-\frac{100}{9}+\frac{36}{9}\right]$$

$$= 3\left(x-\frac{10}{3}\right)^2-\frac{64}{3}$$

2)

a) $x^2 = t \Rightarrow t^2 - 5t + 4 = 0 \Rightarrow (t-4)(t-1) = 0$

$t \quad -4$

$t \quad -1$

$t = 4$ ou $t = 1$

$t = x^2 = 4 \Rightarrow x_1 = -2 \quad x_2 = 2$

$t = x^2 = 1 \Rightarrow x_3 = -1 \quad x_3 = 1$

$S = \{-2; -1; 1; 2\}$

b) $x^2 = t \Rightarrow t^2 + 2t - 8 = 0 \Rightarrow (t+4)(t-2) = 0$

$t \quad +4$

$t \quad -2$

$t = -4$ ou $t = 2$

$t = x^2 = -4 \Rightarrow$ pas de solution réelle !

$t = x^2 = 2 \Rightarrow x_1 = -\sqrt{2} \quad x_2 = \sqrt{2} \Rightarrow S = \{-\sqrt{2}; \sqrt{2}\}$

c) $t = x^2 - 5x \Rightarrow t^2 - 2t - 24 = 0 \Rightarrow (t-6)(t+4) = 0$

$t \quad -6$

$t \quad +4$

$t = 6$ ou $t = -4$

$t = x^2 - 5x = 6 \Rightarrow x^2 - 5x - 6 = 0 \Rightarrow (x-6)(x+1) = 0$

$x \quad -6$

$x \quad +1$

$x_1 = 6 \quad x_2 = -1$

$t = x^2 - 5x = -4 \Rightarrow x^2 - 5x + 4 = 0 \Rightarrow (x-4)(x-1) = 0$

$x \quad -4$

$x \quad -1$

$x_3 = 4 \quad x_4 = 1$

$S = \{-1; 1; 4; 6\}$

d) $\dfrac{x^2}{x-2} = t \Rightarrow t - 4 \cdot \dfrac{1}{t} - 3 = 0 \Rightarrow \dfrac{t^2 - 4 - 3t}{t} = 0 \Rightarrow t^2 - 3t - 4 = 0$

$t \quad -4$

$t \quad +1$

$t = 4 \Rightarrow \dfrac{x^2}{x-2} = 4 \Rightarrow x^2 - 4x + 8 = 0 \quad \Delta = -16 < 0$ pas de solution réelle !

$t = -1 \Rightarrow \dfrac{x^2}{x-2} = -1 \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0 \Rightarrow x_1 = -2 \quad x_2 = 1$

$x \quad +2$

$x \quad -1$

$S = \{-2; 1\}$

2) e) $x-1=\sqrt{2x+1} \Rightarrow (x-1)^2=\sqrt{2x+1}^2 \Rightarrow x^2-2x+1=2x+1$

$$x^2-4x=0$$
$$x(x-4)=0$$
$$S=\{0;4\}$$

3) $S=x_1+x_2=\frac{-b}{a}$ $\quad P=x_1 \cdot x_2=\frac{c}{a}$ $\quad x_1^2+x_2^2=S^2-2P$

$$\left.\begin{array}{l} a=2 \\ b=-5 \\ c=1 \end{array}\right\} \Rightarrow S=\frac{5}{2} \quad P=\frac{1}{2} \quad x_1^2+x_2^2=\frac{25}{4}-2\cdot\frac{1}{2}=\frac{21}{4}$$

$$\frac{x_1+3}{x_2}+\frac{x_2+3}{x_1}=\frac{x_1^2+3(x_1+x_2)+x_2^2}{x_1 \cdot x_2}=\frac{\frac{21}{4}+3\cdot\frac{5}{2}}{\frac{1}{2}}=\frac{51}{2}$$

4) $$\left.\begin{array}{l} a=1 \\ b=-\sqrt{6} \\ c=-m \end{array}\right\} \Rightarrow S=\sqrt{6} \quad P=-m \quad x_1^2+x_2^2=6-2(-m)$$
$$=6+2m$$

$$\frac{1}{x_1^2}+\frac{1}{x_2^2}=\frac{x_1^2+x_2^2}{(x_1 \cdot x_2)^2}=\frac{6+2m}{m^2}=2$$

$2m^2-2m-6=0 \Rightarrow m^2-m-3=0$ → équation du second degré !

$$\Delta_m=(-1)^2-4\cdot1\cdot(-3)=13 \quad \sqrt{\Delta}=\sqrt{13}$$

$$m_1=\frac{1-\sqrt{13}}{2} \qquad m_2=\frac{1+\sqrt{13}}{2}$$

5) a) $$\left.\begin{array}{l} x+y=7 \\ x \cdot y=10 \end{array}\right\} \Rightarrow x_1+x_2=7=\frac{-b}{a}$$
$$x_1 \cdot x_2=10=\frac{c}{a}$$

Soit $a=1$. On a : $x^2-7x+10=0 \Rightarrow x_1=5 \quad x=5$

$x \quad -5$

$x \quad -2 \qquad x_2=2 \quad y=2$

b) $x \cdot y=3 \Rightarrow y=\frac{3}{x}$

$$2x+5\cdot\frac{3}{x}=11 \Rightarrow 2x^2+15=11x \Rightarrow 2x^2-11x+15=0$$

5) b) $x \cdot y = 3 \Rightarrow y = \frac{3}{x}$ $\quad 2x + 5 \cdot \frac{3}{x} = 11$

$$2x^2 + 15 = 11x \Rightarrow 2x^2 - 11x + 15 = 0$$

$$\begin{array}{|cc|} 2x & -5 \\ \hline x & -3 \end{array}$$

$(2x-5)(x-3) = 0$

$x_1 = \frac{5}{2}$ $\qquad x_2 = 3$

$\downarrow$ $\qquad \searrow$

$y_1 = \frac{3}{\frac{5}{2}} = \frac{6}{5}$ $\qquad y_2 = \frac{3}{3} = 1$

$\Rightarrow x = \frac{5}{2}$ et $y = \frac{6}{5}$

ou

$x = 3$ et $y = 1$

c) $y = \frac{-15}{x^2+1} \Rightarrow x^2 + 2 \cdot \frac{-15}{x^2+1} = -2$

$$x^4 + x^2 - 30 = -2x^2 - 2$$

$x^4 + 3x^2 - 28 = 0 \quad t = x^2 \Rightarrow t^2 + 3t - 28 = 0$

$$\begin{array}{cc} t & +7 \\ t & -4 \end{array}$$

$(t+7)(t-4) = 0 \Rightarrow t = -7$ ou $t = 4$

$t = x^2 = -7 \Rightarrow$ pas de solution réelle.

$t = x^2 = 4 \Rightarrow x_1 = -2 \quad x_2 = 2$

$\downarrow \qquad \downarrow$

$y_1 = \frac{-15}{(-2)^2+1} \qquad y_2 = \frac{-15}{2^2+1} = -3$

$y_1 = -3$

$x = -2 \quad y = -3$

ou

$x = 2 \quad y = -3$

6) Rappel: Si $\Delta < 0$, pas de solution réelle

Si $\Delta = 0$, $\sqrt{\Delta} = 0$, $x_1 = x_2 = \frac{-b \mp 0}{2a} = \frac{-b}{2a}$

Si $\Delta > 0$, deux solutions distinctes.

Double solution $\rightarrow x_1 = x_2 = \frac{-b}{a} \Rightarrow \Delta = 0 \Rightarrow b^2 - 4ac = 0$

$(m+3)^2 - 4 \cdot 1(3m+1) = 0 \Rightarrow m^2 + 6m + 9 - 12m - 4 = 0$

$m^2 - 6m + 5 = 0 \Rightarrow (m-5)(m-1) = 0 \Rightarrow m = 5$ ou $m = 1$

$$\begin{array}{cc} m & -5 \\ m & -1 \end{array}$$

7) $\left.\begin{array}{l} a=1 \\ b=2 \\ c=m+1 \end{array}\right\}$ $S=\frac{-b}{a}=-2$ $P=\frac{c}{a}=m+1$

$x_1^2+x_2^2=S^2-2P=4-2(m+1)=2-2m$

Rappel:

$a^3+b^3=(a+b)(a^2-ab+b^2)$

$$\frac{x_2}{x_1^2}+\frac{x_1}{x_2^2}=\frac{x_2^3+x_1^3}{(x_1 . x_2)^2}$$

(x_2^2) (x_1^2)

$$x_1^3+x_2^3=(x_1+x_2)(x_1^2+x_2^2-x_1.x_2)=(-2)(2-2m-m-1)$$

$$=(-2)(1-3m)=6m-2$$

$$\frac{x_1^3+x_2^3}{(x_1.x_2)^2}=\frac{6m-2}{(m+1)^2}=1 \Rightarrow 6m-2=m^2+2m+1$$

$m^2-4m+3=0$ $m=3$ ou $m=1$

m -3

m -1

Attention!

Si on veut résoudre cette équation dans $\mathbb{R}$:

$\Delta>0 \Rightarrow 4-4.1.(m+1)>0 \Rightarrow 4-4m-4>0 \Rightarrow m<0$

Alors, il n'existe aucun réel m qui vérifie cette égalité!

8) $\left.\begin{array}{l} a=1 \\ b=-2 \\ c=k^2-4 \end{array}\right\}$ $S=\frac{-b}{a}=2$ $P=\frac{c}{a}=k^2-4$

$x_1^2+x_2^2=S^2-2P$

$(x_1-x_2)^2=4 \Rightarrow x_1^2+x_2^2-2x_1x_2=4 \Rightarrow S^2-2P-2P=4$

$S^2-4P=4 \Rightarrow 4-4(k^2-4)=4 \Rightarrow 4-4k^2+16=4$

$4k^2=16 \Rightarrow k^2=4 \Rightarrow k=-2$ ou $k=2$